MATH2050B Mathematical Analysis I

Test 1 suggested Solution*

Question 1. (i) Show that if m, $n \in \mathbb{N}$ such that $m < n$ then $m + 1 \leqslant n$.

(ii) Let $A \subseteq \mathbb{R}$ be oder-convex :

$$\left.\begin{array}{l} a_1 < z < a_2 \\ a_1, a_2 \in A, z \in \mathbb{R} \end{array}\right\} \Rightarrow z \in A.$$

Show that if A is not bounded above and is not bounded below then $A = \mathbb{R}$.

(iii) Let $I_n := [a_n, b_n] \subseteq \mathbb{R}, \forall n \in \mathbb{N}$ such that

$$I_{n+1} \subseteq I_n, \quad \forall n \in \mathbb{N}.$$

Using axioms of $\mathbb{R}$ show that $\bigcap_{n\in\mathbb{N}} I_n \neq \varnothing$. Show further that the intersection is a singleton if $\lim_n (b_n - a_n) = 0$.

Solution:

(i) Let

$$\mathbb{N}_m = \{1, 2, \cdots m\} \cup \{m + K : K \in \mathbb{N}\},$$

which is seen to be an inductive subset of $\mathbb{N}$ so equals $\mathbb{N}$. Since n is strictly large than m, we have $n \in \{m + K : K \in \mathbb{N}\}$, which implies that $n \geq m + 1$.

(ii) Suppose A is not bounded above and not bounded below. Let $x \in \mathbb{R}$. Then x is not a lower bound of A, so there exists $a_1 \in A$ such that $a_1 < x$. Similarly, since x is not an upper bound of A, there exists $a_2 \in A$ such that $a_2 > x$. It follows that $x \in A$ due to the fact that A is oder-convex. Thus we have $\mathbb{R} \subseteq A$. On the assumption that $A \subseteq \mathbb{R}$, we conclude that $A = \mathbb{R}$.

(iii) Since $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$, we have

$$a_1 \leq a_2 \leq \ldots \leq a_n \leq b_n \leq \ldots \leq b_2 \leq b_1.$$

Notice that both $\{a_n\}$ and $\{b_n\}$ are bounded below by a_1 and above by b_1. By completeness axiom of $\mathbb{R}$, we can define $a = \sup_n \{a_n\}$ and $b = \inf_n \{b_n\}$. It follows that $a \leq b$, due to the fact that $a_n \leq b_n$ for all $n \in \mathbb{N}$. We see at once that $[a, b] \neq \emptyset$.

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Next we show that $[a,b] = \bigcap_n [a_n, b_n]$. If $x \in [a,b]$, then $a_n \leqslant a \leqslant x$ for all n, since a is an upper bound of $\{a_n\}$, and similarly we get $x \le b_n$ for all $n \in \mathbb{N}$. Hence that $x \in \bigcap_n [a_n, b_n]$.

Conversely, suppose $x \in \bigcap_n [a_n, b_n]$. Then we have $a_n \le x \le b_n$ for all n. Thus x is an upper bound of $\{a_n\}$, and $a \le x$ (as a being the smallest upper bound of $\{a_n\}$). By a similar argument, we can see that $x \le b$, so $x \in [a,b]$. Therefore we can conclude that $[a,b] = \bigcap_n [a_n, b_n]$.

Suppose additionally that $\lim_n (b_n - a_n) = 0$. Then for any $\epsilon > 0$, pick $N \in \mathbb{N}$ such that $b_N - a_N < \epsilon$. It follows from the preceding paragraph that $b - a \le b_N - a_N < \epsilon$. Since ϵ is arbitrary, we have $b - a = 0$, and $\{a\} = \{b\} = \bigcap_n [a_n, b_n]$.

Question 2. *(Not to use on any theorem (limits))* In the terminology of $\varepsilon - N$, show that

(i) If $\lim_n x_n = 3$ and $\lim_n y_n = -2$ then

$$\lim_n (x_n y_n) = -6 \quad \text{and}$$
$$\lim_n \frac{x_n^2 + 3}{x_n - 2} = 12.$$

(ii) If $\lim_n z_n = l \in \mathbb{R}$ and $z_n \geqslant 0, \forall n \in \mathrm{N}$ then $l \geqslant 0$ and $\lim_n \sqrt{z_n} = \sqrt{l}$.

Solution:

(i) Fix $\epsilon > 0$. Since $\lim_n x_n = 3$, there exists $N_1(\epsilon) \in \mathbb{N}$ such that for any $n \ge N_1(\epsilon)$, we have $|x_n - 3| < \frac{\epsilon}{6}$. Similarly, since $\lim_n y_n = -2$, there exists $N_2(\epsilon) \in \mathbb{N}$ such that for any $n \ge N_2(\epsilon)$,

$$|y_n - (-2)| < 1 \quad \text{and} \quad |y_n - (-2)| < \frac{\epsilon}{6},$$

which implies that for all $n \ge N_2(\epsilon)$,

$$|y_n| < 3 \text{ and } |y_n - (-2)| < \frac{\epsilon}{6}.$$

Let $N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon)\}$, it follows that for any $n \ge N(\epsilon)$,

$$\begin{aligned} |x_n y_n - (-6)| &= |x_n y_n - 3y_n + 3y_n - (-6)| \\ &\le |x_n y_n - 3y_n| + |3y_n - (-6)| \\ &\le |y_n| \cdot |x_n - 3| + 3|y_n + 2| \\ &\le 3 \cdot \frac{\epsilon}{6} + 3 \cdot \frac{\epsilon}{6} \\ &= \epsilon, \end{aligned}$$

which yields that $\lim_n (x_n y_n) = -6$.

Next we show that $\lim_n \frac{x_n^2 + 3}{x_n - 2} = 12$.

Since $\lim_n x_n = 3$, there exists $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$,

$$|x_n - 3| < \frac{1}{2} \quad \text{i.e.} \quad \frac{5}{2} < x_n < \frac{7}{2},$$

which implies that $x_n - 2 > \frac{1}{2}$ and $|x_n| < \frac{7}{2}$.

Fix $\epsilon > 0$. Since $\lim_n x_n = 3$, there exists $N_2(\epsilon) \in \mathbb{N}$ such that for any $n \geq N_2(\epsilon)$,

$$|x_n - 3| < \frac{\epsilon}{25}.$$

Let $N(\epsilon) = \max\{N_1, N_2(\epsilon)\}$. Then for any $n \geq N(\epsilon)$,

$$\begin{aligned}
\left|\frac{x_n^2 + 3}{x_n - 2} - 12\right| &= \left|\frac{x_n^2 - 12x_n + 27}{x_n - 2}\right| \\
&= \left|\frac{(x_n - 3)(x_n - 9)}{x_n - 2}\right| \\
&\leq 2|x_n - 3|(|x_n| + 9) \\
&\leq 2\frac{\epsilon}{25} \cdot \frac{25}{2} \\
&= \epsilon,
\end{aligned}$$

that is, $\lim_n \dfrac{x_n^2 + 3}{x_n - 2} = 12$.

(ii) We first show that $\ell \geq 0$. Suppose on the contrary that $\ell < 0$. Since $\lim_n z_n = \ell$, there exists $N \in \mathbb{N}$ so that for any $n \geq N$, we have $|z_n - \ell| < \dfrac{|\ell|}{2}$, which yields that $z_n < \ell + \dfrac{|\ell|}{2} < 0$. This contracts with the assumption.

Next we show that $\lim_n \sqrt{z_n} = \sqrt{\ell}$.

Case 1:

$$|z_n| < \epsilon^2 \quad \text{i.e.} \quad |\sqrt{z_n}| < \epsilon,$$

which yields that $\lim_n \sqrt{z_n} = 0$.

Case 2: $\lim_n z_n = \ell > 0$. For any $\epsilon > 0$ there exists $N_2(\epsilon) \in \mathbb{N}$ so that for any $n \geq N_2(\epsilon)$,

$$-\frac{\ell}{2} < z_n - \ell < \frac{\ell}{2} \quad \text{and} \quad |z_n - \ell| < \frac{3\sqrt{\ell}}{2}\epsilon,$$

which also implies that $|z_n| > \frac{\ell}{4}$ since $\ell > 0$.

Thus for any $n \geq N_2(\epsilon)$,

$$|\sqrt{z_n} - \sqrt{\ell}| = \left|\frac{z_n - \ell}{\sqrt{z_n} + \sqrt{\ell}}\right| < \left|\frac{z_n - \ell}{\sqrt{\ell}/2 + \sqrt{\ell}}\right| = \frac{2}{3\sqrt{\ell}}|z_n - \ell| < \frac{2}{3\sqrt{\ell}}\frac{3\sqrt{\ell}}{2}\epsilon = \epsilon.$$

Therefore we have $\lim_n \sqrt{z_n} = \sqrt{\ell}$.

Question 3. State (without proof) the Bolzano-Weierstrass Theorem and hence show that a sequence (x_n) is convergent if it is Cauchy.

Solution:

Bolzano-Weierstrass Theorem: A bounded sequence of real numbers has a convergent subsequence.

Now we show that if $\{x_n\}$ is a Cauchy sequence, then $\{x_n\}$ is bounded. Since $\{x_n\}$ is Cauchy, there exists $N \in \mathbb{N}$, such that for any $n \geq N$,

$$|x_n - x_N| \leq 1,$$

that is, $x_N - 1 \leq x_n \leq x_N + 1$ for any $n \geq N$. Denote $a = \min\{x_1, \dots, x_{N-1}, x_N, x_N - 1\}$ and $b = \max\{x_1, \dots, x_{N-1}, x_N, x_N + 1\}$, it is clear that $\{x_n\}$ is bounded below by a and bounded above by b.

It follows from Bolzano-Werierstrass Theorem that there exists a convergent subsequence $\{x_{n_k}\}$. Suppose $\lim_{k\to\infty} x_{n_k} = x$. Fix $\epsilon > 0$, there exists $k_1 \in \mathbb{N}$ such that for any $k \geq k_1$,

$$|x_{n_k} - x| < \frac{\epsilon}{2}.$$

On the other hand, since $\{x_n\}$ is Cauchy, there is $N_1 \in \mathbb{N}$ so that for any $m, n \geq N_1$,

$$|x_m - x_n| < \frac{\epsilon}{2}.$$

Note that $\{n_k\}_k$ is an increasing sequence tending to infinity, there exists $k_2 \geq k_1$ such that $n_k \geq N_1$ for all $k \geq k_2$.

Denote $N_2 = n_{k_2}$, then for any $m \geq N_2$, we have

$$|x_m - x| < |x_m - x_{n_{k_2}}| + |x_{n_{k_2}} - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $\lim_{n\to\infty} x_n = x$.